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AN EQUIVALENCE FOR THE EMBEDDINGS OF CELLS IN A 3-MANIFOLD

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1. **Introduction.** In studying the embedding of a topological cell C in a manifold M, it is useful to know whether a lower dimensional cell C' has an embedding (in M) which is similar, under some reasonable definition of the word, to that of C. In this paper we show that if C is an i-cell in the interior of a 3-manifold M and j is a positive integer less than i, then there exist a j-cell D in M and a map f of M onto itself such that $f(C)=D$ and f is a homeomorphism of $M-C$ onto $M-D$; furthermore, the map f restricted to C acts formally like a projection map, and f is the identity map outside a preassigned neighborhood of C.

In case that C is a cell in the interior of a 3-manifold M and D is a cell in Bd C such that C is locally tame at points of $C-D$, then well-known techniques provide such a map collapsing C onto D. Otherwise, the facts established about this problem concern the related question for decomposition spaces; namely, if C is a cell in Int M, is there a lower dimensional cell C' in Int M such that M/C and M/C' are homeomorphic? For example, Armentrout [3] and Meyer [17] have shown that for special embeddings of a 3-cell C in E^3 there is an arc A in E^3 such that E^3/A is homeomorphic to E^3/C. Armentrout, Lininger, and Meyer [4] have also proved that if C is a tamely finnable 3-cell in E^3, there is a 2-cell D in E^3 such that E^3/D and E^3/C are homeomorphic. Corollary 5 is an extension of these results to arbitrary embeddings of a 3-cell C in E^3.

The main results of this paper are stated in §3, although the proof of one of the theorems, involving some intricate geometry and epsilonics, is delayed until §4. In §5 we discuss extensions of the results of §3 to embeddings of a cube with handles in the interior of a 3-manifold.

2. **Definitions and notation.** We use Δ_1, Δ_2, and Δ_3 to denote standard 1, 2, and 3-cells defined by $\Delta_3=\{(x, y, z) \mid x^2+y^2+z^2 \leqq 1\}$, $\Delta_2=\{(x, y, 0) \mid x^2+y^2 \leqq 1\}$ and $\Delta_1=\{(x, 0, 0) \mid -1 \leqq x \leqq 1\}$. We use ρ to denote the projection map of Δ_3 onto Δ_2 taking (x, y, z) to $(x, y, 0)$ and π to denote the projection map of Δ_2 onto Δ_1 taking $(x, y, 0)$ to $(x, 0, 0)$.

Let M be a 3-manifold (possibly with boundary). We use d to denote a metric on M. If A is a subset of M and δ is a positive number, then $N(A, \delta)$ denotes the set

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of points of M whose distance from A is less than δ, and Diam A denotes the diameter of A.

An *n-frame* G_n is the union of n arcs $A_1, \ldots, A_n$ with a distinguished point p such that p is an endpoint of each A_i and $A_i \cap A_j = p$ for $i \neq j$. The boundary of G_n, denoted Bd G_n, is $\bigcup$ (Bd $A_i - p$).

If X is a metric space and C is a closed subset of X, then X/C designates the decomposition space associated with the upper semicontinuous decomposition of X whose only nondegenerate element is C.

The unit interval [0, 1] is denoted by I.

3. **Equivalent embeddings.** McMillan [16] has shown that any cell embedded in the interior of a 3-manifold has a neighborhood that can be embedded in E^3. Consequently, in the proofs of all the theorems in this section we may assume that the 3-manifold M is E^3.

THEOREM 1. *Suppose that K is a 3-cell in the interior of a 3-manifold M and that U is an open subset of M with $K \subset U$. Then there exist a map f of M onto M, a homeomorphism g of Δ_3 onto K, and a homeomorphism h of Δ_2 onto $f(K)$ such that*

(1) *f is a homeomorphism of $M-K$ onto $M-f(K)$,*

(2) *$f|M-U=$ identity,*

(3) *$fg=h\rho$.*

Proof. Let J be a simple closed curve in Bd K, and let D_1 and D_2 be the disks in Bd K bounded by J. By the techniques of [11] there is a map f of $M-$ Int K onto M such that

(1) f is a homeomorphism of $M-K$ onto $M-f(D_1)$,

(2) $f|M-U=$ identity,

(3) $f(D_1)=f(D_2)$,

(4) $f|D_i$ is a homeomorphism ($i=1, 2$).

The map h may be any homeomorphism of Δ_2 onto $f(D_1)$, and it is then a simple matter to extend f over all of M and to obtain the required homeomorphism g.

The proof of the following theorem is discussed in §4.

THEOREM 2. *Suppose that F is a 2-cell in the interior of a 3-manifold M and that U is an open subset of M with $F \subset U$. Then there exist a map f of M onto M, a homeomorphism g of Δ_2 onto F, and a homeomorphism h of Δ_1 onto $f(F)$ such that*

(1) *f is a homeomorphism of $M-F$ onto $M-f(F)$,*

(2) *$f|M-U=$ identity,*

(3) *$fg=h\pi$.*

THEOREM 3. *Suppose that K is a 3-cell in the interior of a 3-manifold M and that U is an open subset of M with $K \subset U$. Then there exist a map f of M onto M, a homeomorphism g of Δ_3 onto K, and a homeomorphism h of Δ_1 onto $f(K)$ such that*

(1) *f is a homeomorphism of $M-K$ onto $M-f(K)$,*

(2) *$f|M-U=$ identity,*

(3) *$fg=h\pi\rho$.*

Proof. Apply Theorem 1 to obtain a map f_1 of M onto itself that takes K onto a 2-cell. Then, by Theorem 2, there exist a map f_2 of M onto itself, a homeomorphism g_2 of Δ_2 onto $f_1(K)$, and a homeomorphism h of Δ_1 onto $f_2f_1(K)$ such that f_2 is a homeomorphism of $M-f_1(K)$ onto $M-f_2f_1(K)$, $f_2|M-f_1(U)=\text{identity}$, and $f_2g_2=h\pi$. Define $f|M-\text{Int } K=f_2f_1|M-\text{Int } K$, and define g' to be a homeomorphism of Bd Δ_3 onto Bd K such that $f_1g'=g_2\rho$. Hence, $fg'=f_2f_1g'=f_2g_2\rho=h\pi\rho$. To complete the proof, let g be a homeomorphism of Δ_3 onto K extending g', and let $f|\text{Int } K=h\pi\rho g^{-1}|\text{Int } K$.

Corollary 4. *If K is a cell in the interior of a 3-manifold M, then there exists an arc A in M such that $M-A$ is homeomorphic to $M-K$.*

Corollary 5. *If K is a cell in the interior of a 3-manifold M, then there exists an arc A in M such that M/A is topologically M/K. Furthermore, if K is not a 1-cell then there exists a disk D in M such that M/D is topologically M/K.*

From Corollary 5 and the theorem by Andrews and Curtis [2] we obtain another proof of a result due to Bryant [9].

Corollary 6. *If K is a cell in E^3, then $E^3/K\times E^1$ is homeomorphic to E^4.*

The next two corollaries follow from Corollary 5 and Kwun's extension [14] of the Andrews and Curtis theorem.

Corollary 7. *If K_1 and K_2 are cells in E^3, then $E^3/K_1\times E^3/K_2$ is homeomorphic to E^6.*

Corollary 8. *If K is a cell in E^3 and α is an arc in E^n, then $E^n/\alpha\times E^3/K$ is homeomorphic to E^{n+3}.*

Remarks. The tameness of a cell is preserved by this squeezing process, for if K is a 3-cell in Int M and f is a map of M onto itself that satisfies the conclusions of Theorem 1, then well-known results such as Bing's 1-ULC Criterion [6] imply that $f(K)$ is tame; if K is a tame cell in Int M and f is a map of M onto itself that satisfies the conclusions of either Theorem 2 or Theorem 3, then Theorem 1 of [8] implies that $f(K)$ is tame.

However, if K is a wild cell, the images of K under two different projection maps may be inequivalently embedded. For example, if K is the 3-cell described in §2 of [1], there is a map f_1 of E^3 onto itself satisfying the conclusions of Theorem 3 that collapses K onto the arc W of points in Bd K where Bd K is wild, and, therefore, $f_1(K)$ is a wild arc; there is another map f_2 of E^3 onto itself satisfying the conclusions of Theorem 3 such that $f_2(W)$ is a point, and it can be shown, using properties of this embedding of K, that $f_2(K)$ is a tame arc.

The following result, a converse to Theorem 1, indicates that near each disk F in E^3 there is a 3-cell K that projects (in the sense of Theorem 1) onto the disk. Of course, it may be impossible for K to contain the disk; therefore, the set of points moved by the projection map cannot be restricted to neighborhoods arbitrarily close to K(2).

THEOREM 9. *If h is a homeomorphism of* Δ_2 *onto a* 2-*cell F in the interior of a* 3-*manifold M and U is an open subset of M containing* Int F, *then there exist a homeomorphism g of* Δ_3 *onto a* 3-*cell K in* Bd $F \cup U$ *and a map f of M onto M such that*

(1) *f is a homeomorphism of $M-K$ onto $M-F$,*

(2) *$f|M-U=$ identity,*

(3) *$fg=h\rho$.*

Proof. There is a deformation retraction r_t $(0 \leqq t \leqq 1)$ of E^3 onto F. Let V be a connected open subset of U containing Int F such that if $x \in V$ then $r_t(x) \notin$ Bd F $(0 \leqq t \leqq 1)$. It can be shown that $V-$Int F is the disjoint union of two open sets V_1 and V_2, each of which contains F in its closure.

It follows from techniques of [13] or [15], that for $i=1, 2$, there exists a homeomorphism f_i of Cl V_i into Cl V such that $f_i|$Bd V_i-Int $F=$identity, f_i(Int F) is locally tame from $V-f_i$(Cl V_i), and f_1(Cl V_1) $\cap f_2$(Cl V_2)$=$Bd F.

Let S be the 2-sphere $f_1(F) \cup f_2(F)$. Hence, by the construction, S is locally tame from Int S at points of $S-$Bd F. In order to show that S is locally tame from Int S at points of Bd F, it is sufficient to show that if $x \in$ Bd F and $\varepsilon>0$, then there exists a positive number δ such that each simple closed curve in $N(x, \delta) \cap$ Int S can be shrunk to a point in an ε-subset of E^3-Bd F.

Let α be a positive number small enough that closed α-subsets of Int F lie in $\varepsilon/3$-disks in Int F, and let $x \in$ Bd F. Since $r_t(x)=x$ for $0 \leqq t \leqq 1$, there is a positive number δ such that Diam $\bigcup_t r_t(N(x, \delta))<\alpha$. If J is a simple closed curve in $N(x, \delta) \cap$ Int S, then $J \subset V$ and $r_t(J) \cap$ Bd $F=\varnothing$ $(0 \leqq t \leqq 1)$. It follows that J can be shrunk to a point in an ε-subset of Int $F \cup (\bigcup_t r_t(J))$.

Therefore, S is locally tame from Int S, and we let K be Cl (Int S). Let g_1 be a homeomorphism of Bd Δ_3 onto S such that $h\rho g_1^{-1}(x)=f_i^{-1}(x)$ for $x \in f_i(F)$ $(i=1, 2)$, and let g be a homeomorphism of Δ_3 onto K extending g_1. The desired map f is given by the rule

$$\begin{aligned} f(x) &= x && \text{if } x \in E^3-(K \cup f_1(V_1) \cup f_2(V_2)) \\ &= f_1^{-1}(x) && \text{if } x \in f_1(V_1) \\ &= f_2^{-1}(x) && \text{if } x \in f_2(V_2) \\ &= h\rho g^{-1}(x) && \text{if } x \in K. \end{aligned}$$

4. **Proof of Theorem 2.** We need the following definition.

DEFINITION. A disk D is *normally situated* in a surface S if D either lies in Int S or intersects Bd S in an arc. A Sierpiński curve is *normally situated* in a surface if the closures of the components of its complement are normally situated disks.

(2) Theorem 9 was communicated to the first named author by Robert F. Craggs, whose more detailed proof will appear shortly.

The following lemma is due to Craggs [10, Lemma 5.1].

LEMMA 0. *Suppose that M is a 3-manifold, D is a disk in M, and ε is a positive number. Then there is a tame Sierpiński curve X in D which is normally situated in D such that each component of $D-X$ has diameter less than ε.*

Furthermore if $\{X_j\}$ is a finite collection of sets, each of which is either a tame arc in D or a tame Sierpiński curve normally situated in D, then X may be chosen so that $\bigcup_j X_j$ is contained in the inaccessible part of X.

LEMMA 1. *Suppose $0\leqq r<t<s\leqq 1$, $0<r_1<r_2<\cdots<r_k<1$, g is a homeomorphism of $[r, s]\times I$ into E^3 such that $g([r, s]\times r_1), \dots, g([r, s]\times r_k)$, and $g(t\times I)$ are tame, and U is an open set in E^3 containing $g(t\times I)$. Then there exist positive numbers $r<u_k<u_{k-1}<\cdots<u_1<t<v_1<\cdots<v_k<s$, a homeomorphism λ of $[r, s]\times I$ onto itself, and a map β of E^3 onto E^3 such that*

(1) *$\lambda|\{r, t, s\}\times I=$ identity,*

(2) *$\beta|E^3-U=$ identity,*

(3) *$\beta g(t\times I)$ is a point in U,*

(4) *$\beta|E^3-g(t\times I)$ is a homeomorphism onto $E^3-\beta g(t\times I)$.*

Furthermore, if $A_i=(u_i\times[r_i, 1])\cup([r, u_i]\times r_i)$, $B_i=(v_i\times[r_i, 1])\cup([v_i, s]\times r_i)$, δ is the diameter of the largest component of $g(([r, s]\times I)-[(I\times\{r_1, \dots, r_k\})\cup(t\times I)])$, δ' is the diameter of the largest component of $([r, s]\times I)-[(I\times\{r_1, \dots, r_k\})\cup(t\times I)]$, and C is a component of $([r, s]\times I)-\bigcup_i(A_i\cup B_i\cup(t\times I))$ then

(5) *$d(x, \lambda(x))<\delta'$,*

(6) Diam $\beta g\lambda(C)<5\delta$,

(7) *the arcs $g\lambda(A_i)$ and $g\lambda(B_i)$ are tame.*

Proof. Using Lemma 0 it is straightforward to show that there exist a Sierpiński curve $X\subset[r, s]\times I$ and a homeomorphism h of E^3 onto E^3 which moves no point outside a compact subset of E^3 such that

(8) X is normally situated in $[r, s]\times I$,

(9) $[r, s]\times r_i$ and $t\times I$ lie in the inaccessible part of X,

(10) $hg(X)$ lies in the xy-plane and the outer boundary component of $hg(X)$ is the rectangle with vertices $(\pm 1, 0, 0)$ and $(\pm 1, k+1, 0)$,

(11) $hg(t\times I)=\{(x, y, z) \mid x=0, 0\leqq y\leqq k+1, z=0)\}$,

(12) $hg(x\times r_i)=((x-t)/(s-t), i, 0)$ for $t\leqq x\leqq s$,
$=((x-t)/(t-r), i, 0)$ for $r\leqq x\leqq t$,

(13) for any positive number ε there is a number ε' such that $0<\varepsilon'<\varepsilon$ and the pair of arcs $\{(x, y, z) \mid |x|=\varepsilon', 0\leqq y\leqq k+1, z=0)\}$ lie in $hg(X)$.

Using the uniform continuity of h^{-1}, we find a positive number α small enough that α-subsets of E^3 go to δ-sets under h^{-1}.

If $t_0, t_1, t_2, \dots, t_{k+1}$ is a sequence of numbers such that $1>t_0>\cdots>t_{k+1}=0$, then the following cells are associated with the t_i's: for $i=0, \dots, k$, D_i is the straight line segment from $(t_i, k+1-i, 0)$ to $(t_i, k+1, 0)$; F_i is the line segment from $(t_i, k+1-i, 0)$ to $(t_{i+1}, k-i, 0)$; P_i is the 2-cell in the xy-plane bounded by the quadrilateral with vertices $(t_i, k+1, 0)$, $(t_i, k+1-i, 0)$, $(t_{i+1}, k-i, 0)$ and $(t_{i+1}, k+1, 0)$; R_i is the 2-cell in the xy-plane bounded by the quadrilateral with vertices $(1, k+1-i, 0)$, $(1, k-i, 0)$, $(t_i, k+1-i, 0)$ and $(t_{i+1}, k-i, 0)$; G_i is the 2-cell in the xy-plane bounded by the quadrilateral with vertices $(\pm t_i, k+1-i, 0)$ and $(\pm t_{i+1}, k-i, 0)$; D_i', F_i', P_i', and R_i' are the mirror images of the cells D_i, F_i, P_i, and R_i, respectively, on the other side of the plane $x=0$; $E_i=P_i \cup R_i$; $E_i'=P_i' \cup R_i'$; $N_i=\{(x, y, z) \mid t_{i+1}^2-x^2 \leqq (y-k-1)^2 \leqq t_i^2-x^2,\ y \geqq k+1,\ z=0\}$; T_i and H_i are the solids of revolution obtained by revolving $N_i \cup P_i \cup P_i'$ and G_i, respectively, about the y-axis; and $M=\bigcup T_i$.

Since the components of $([r, s]\times I)-X$ form a null sequence, it follows from (8), (9), and (13) that there is a sequence of numbers $t_0, t_1, \ldots, t_{k+1}$ such that

(14) $\alpha/2 > t_0 > t_1 > \cdots > t_{k+1}=0$,

(15) the arcs D_i and D_i' lie in $hg(X)$,

(16) if K is a component of $([r, s]\times I)-X$ such that $hg(\mathrm{Bd}\ K)\subset E_i \cup E_i'$, then $hg(K) \cap (\bigcup \{T_j \mid |j-i|>1\})=\varnothing$,

(17) $M \subset h(U)$.

There is a map μ of E^3 onto E^3 such that

(18) $\mu | E^3-M=\text{identity}$,

(19) $\mu(T_0)=\mathrm{Cl}\,(M-\bigcup_{i=1}^k H_i)$,

(20) $\mu(T_i)=H_i$ for $i=1, \ldots, k$,

(21) $\mu hg(t\times I)=(0, 0, 0)$,

(22) $\mu | E^3-hg(t\times I)$ is a homeomorphism onto $E^3-(0, 0, 0)$.

The action of the map μ in the xy-plane is illustrated in Figure 1.

The required map $\beta=h^{-1}\mu h$. The numbers u_i and v_i are given by $(u_i, 0)=g^{-1}h^{-1}(-t_{k+1-i}, 0, 0)$ and $(v_i, 0)=g^{-1}h^{-1}(t_{k+1-i}, 0, 0)$, and λ is a homeomorphism of $[r, s]\times I$ onto $[r, s]\times I$ such that

$$\begin{aligned}\lambda \mid (\textstyle\bigcup_i ([r, s]\times r_i)) \cup (t\times I) &= \text{identity},\\ \lambda(u_i\times[r_i, 1]) &= g^{-1}h^{-1}(D'_{k+1-i}),\\ \lambda(v_i\times[r_i, 1]) &= g^{-1}h^{-1}(D_{k+1-i}).\end{aligned}$$

It is straightforward to check that conditions (1) through (7) are satisfied.

LEMMA 2. *Suppose E is a disk topologically embedded in E^3, g is a map of I^2 $(=I\times I)$ onto E such that $g(0\times I)$ and $g(1\times I)$ are distinct points in* Bd *E, $g|(0, 1)\times I$ is a homeomorphism onto $E-g(\{0, 1\}\times I)$, U is an open set in E^3 containing $E-g(\{0, 1\}\times I)$, and $\varepsilon>0$. Then there exist a partition $\{t_i\}$ of I, an ε-homeomorphism α of I^2 onto itself, and a map β of E^3 onto E^3 such that*

(1) $0=t_0<t_1<\cdots<t_n=1$ *and* $t_i-t_{i-1}<\varepsilon$,

(2) $\alpha|\{0, 1\}\times I=$ *identity*,

(3) $\beta|E^3-U=$ *identity*,

(4) $\beta g\alpha(t_i\times I)$ *is a point in U for* $i=0, 1, \ldots, n$,

(5) $\beta g\alpha(t_i\times I)\neq\beta g\alpha(t_j\times I)$ *if* $i\neq j$,

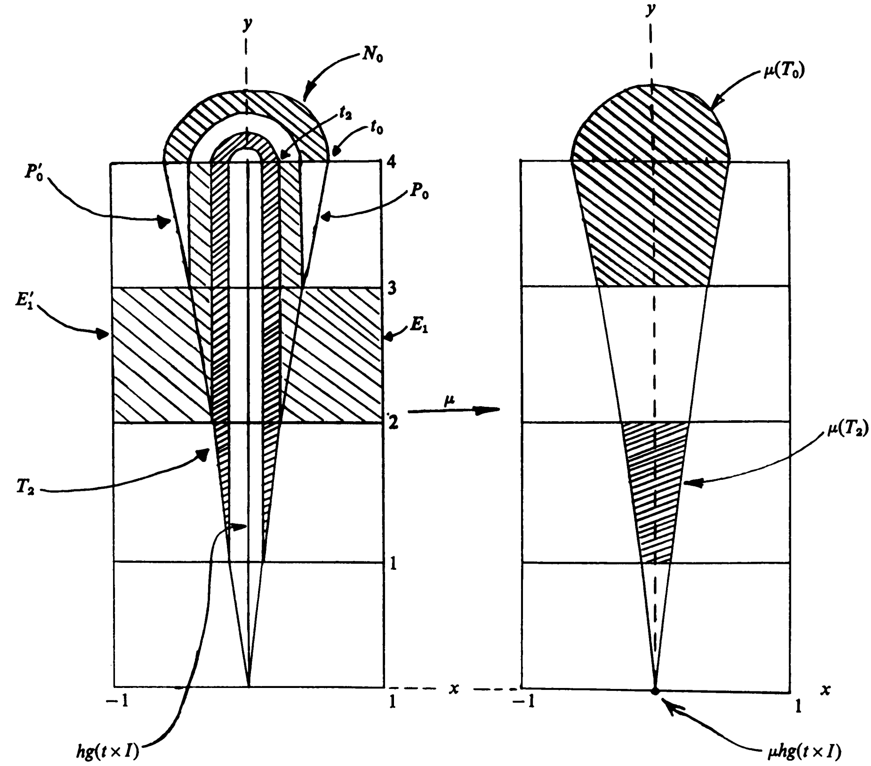

FIGURE 1

(6) $\beta|E^3 - \bigcup_{i=1}^{n-1} g\alpha(t_i \times I)$ *is a homeomorphism onto* $E^3 - \bigcup_{i=1}^{n-1} \beta g\alpha(t_i \times I)$,

(7) Diam $\beta g\alpha([t_{i-1}, t_i] \times I) < \varepsilon$ *for* $i = 1, \ldots, n$.

Proof. The argument is given in two steps. In Step 1 the disk $g(I^2) = E$ is sliced into thin "vertical" strips by arcs, and these arcs are squeezed to points using Lemma 1. In Step 2 we find other arcs slicing the images of these strips into *small* disks. Enough care must be exercised in Step 1 to insure that the arcs of Step 2 can be realized as images of approximately vertical segments in I^2.

Step 1. Using Lemma 0 it is straightforward to show that there exist an $\varepsilon/3$-homeomorphism α_1 of I^2 onto I^2, and an integer $k > 3/\varepsilon$ such that

(8) $\alpha_1|\{0, 1\} \times I = \text{identity}$,

(9) $g\alpha_1((i/2k) \times I)$ is tame for $i = 1, \ldots, (2k-1)$,

(10) $g\alpha_1([1/2k, (2k-1)/2k] \times i/2k)$ is tame for $i = 1, \ldots, (2k-1)$,

(11) Diam $g\alpha_1([0, 1/k] \times I) < \varepsilon$ and Diam $g\alpha_1([(k-1)/k, 1] \times I) < \varepsilon$,

(12) Diam $g\alpha_1([i/2k, (i+1)/2k] \times [j/2k, (j+1)/2k]) < \varepsilon/10$ for $i, j = 1, \ldots, (2k-2)$.

Let $B_i = (i/k) \times I$ and let $U_1, U_2, \ldots, U_{k-1}$ be disjoint open sets in E^3 such that

$$g\alpha_1(B_i) \subset U_i \subset U \quad \text{and} \quad g\alpha_1(([0, (2i-1)/2k] \cup [(2i+1)/2k, 1]) \times I) \cap U_i = \varnothing.$$

Define a homeomorphism μ of I^2 onto I^2 by $\mu((x, y)) = (x, 1-y)$. In the statement of Lemma 1 take $U=U_i$, $t=i/k$, $r_j=j/2k$, $(j=1,\ldots,2k-1)$, and then apply Lemma 1 to the homeomorphism $g\alpha_1 \mid [(2i-1)/2k, (2i+1)/2k]\times I$ if i is an odd integer and to the homeomorphism $g\alpha_1\mu \mid [2i-1/2k, 2i+1/2k]\times I$ if i is an even integer $(i=1,2,\ldots,k-1)$, thus obtaining maps $\lambda_1,\ldots,\lambda_{k-1}$, $\beta_1,\ldots,\beta_{k-1}$ satisfying the conclusions of Lemma 1.

The homeomorphisms λ_1, $\mu\lambda_2\mu^{-1}$, λ_3, $\mu\lambda_4\mu^{-1}$, $\lambda_5,\ldots$ are pieced together to obtain a homeomorphism α_2 of $I\times I$ onto itself. Also, the maps $\beta_1,\ldots,\beta_{k-1}$ are pieced together to form a map β' of E^3 onto E^3. Note that

(13) $\alpha_2|\bigcup B_i=\text{identity}$,

(14) $d(x, \alpha_2(x))<\varepsilon/3$,

(15) $\beta' g\alpha_1(B_i)$ is a point in U_i,

(16) $\beta'|E^3-\bigcup U_i=\text{identity}$,

(17) $\beta'|E^3-g\alpha_1(\bigcup B_i)$ is a homeomorphism onto $E^3-\beta' g\alpha_1(\bigcup B_i)$.

Step 2. It follows that there are numbers $u(i,1),\ldots,u(i,2k-1)$ and $v(i,1),\ldots,v(i,2k-1)$ satisfying

$$i/k < u(i,1) < \cdots < u(i,2k-1) < (2i+1)/2k < v(i,1) < \cdots < v(i,2k-1) < (i+1)/k$$

such that

(18) $g\alpha_1\alpha_2(A_{ij})$ is tame, and

(19) Diam $\beta' g\alpha_1\alpha_2(C)<\varepsilon$ for each component C of

$$I^2-\Big(\bigcup_{i,j}(A_{ij}\cup B_i)\Big),$$

where the A_{ij}'s are arcs defined by

$$A_{ij} = (u(i,j)\times[r_j, 1])\cup([u(i,j), v(i,j)]\times r_j)\cup(v(i,j)\times[0, r_j]) \quad \text{if } i \text{ is an odd integer}$$
$$= (u(i,j)\times[0, 1-r_j])\cup([u(i,j), v(i,j)]\times(1-r_j))\cup(v(i,j)\times[1-r_j, 1]) \quad \text{if } i \text{ is an even integer.}$$

There is a homeomorphism α_3 of I^2 onto I^2 such that

(20) $d(x, \alpha_3(x))<\varepsilon/3$,

(21) $\alpha_3|B_i=\text{identity}$,

(22) $\alpha_3|([0, 1/k]\cup[(k-1)/k, 1])\times I=\text{identity}$,

(23) $\alpha_3(t_{ij}\times I)=A_{ij}$ where $i/k<t_{i1}<\cdots<t_{i,k-1}<i+1/k$.

The required homeomorphism $\alpha=\alpha_1\alpha_2\alpha_3$; the map β is β' followed by a map of E^3 onto E^3 that is the identity outside a small neighborhood of $\bigcup_{i,j} g\alpha(t_{ij}\times I)$ and takes the tame arcs $g\alpha(t_{ij}\times I)$ to distinct points. The map β' is illustrated in Figure 2.

The following is a stronger version of Theorem 2.

THEOREM 2′. *Suppose that F is a 2-cell in the interior of a 3-manifold M, g_0 is a homeomorphism of Δ_2 onto F, U is an open subset of M containing $F-g_0(\text{Bd}\,\Delta_1)$, and*

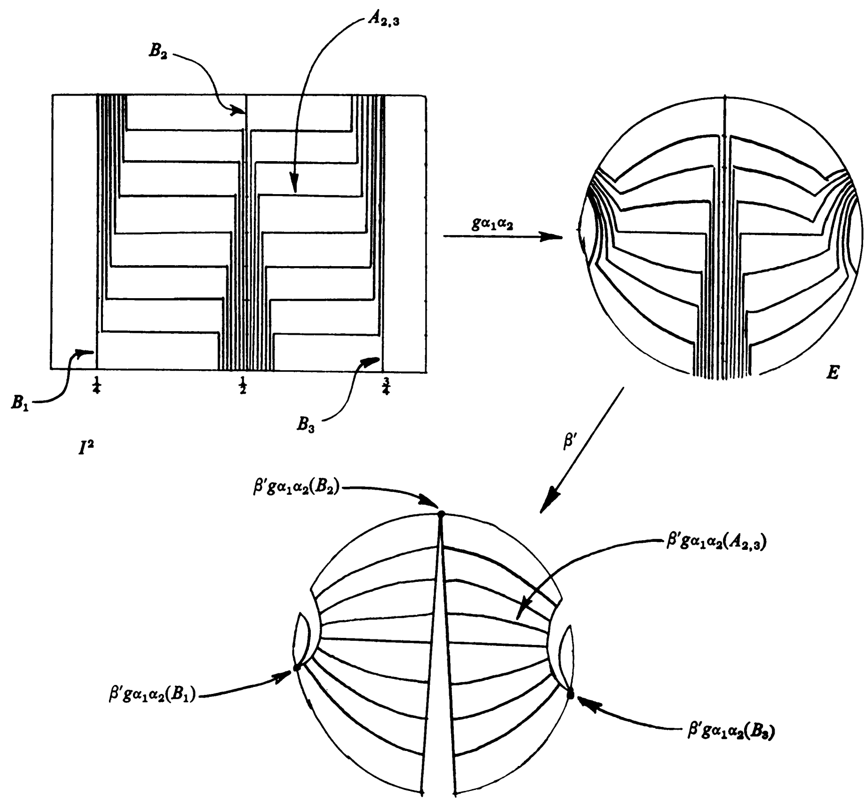

FIGURE 2

$\varepsilon > 0$. *Then there exist a map f of M onto M, a homeomorphism g of Δ_2 onto F, and a homeomorphism h of Δ_1 onto $f(F)$ such that*

(1) *f is a homeomorphism of $M - F$ onto $M - f(F)$,*

(2) *$f|M - U = \text{identity}$,*

(3) *$fg = h\pi$,*

(4) *$d(g_0, g) < \varepsilon$,*

(5) *$g|\text{Bd } \Delta_1 = g_0|\text{Bd } \Delta_1$.*

Proof. Let P be a map of I^2 onto Δ_2 so that $P(0 \times I) = (-1, 0, 0)$, $P = (1 \times I) = (1, 0, 0)$, $P|(0, 1) \times I$ is a homeomorphism onto $\Delta_2 - \{(-1, 0, 0), (1, 0, 0)\}$ and $P(t \times I) = \pi^{-1}(r)$ for some $r \in \Delta_1$. The maps g and f are obtained as limits of sequences of homeomorphisms $\{g_n\}$ and maps $\{f_n\}$ respectively.

Let $U_0, U_1, \ldots$ be a sequence of bounded open sets in E^3 such that $U \supset U_0 \supset U_1 \cdots$ and $\bigcap U_i = F$, and let $\varepsilon_0, \varepsilon_1, \ldots$ be a sequence of positive numbers such that

(6) $\sum_0^\infty \varepsilon_i < \varepsilon$.

Let f_0 be the identity map of E^3 onto E^3, and let V_0 be a connected open set so that $F \subset V_0 \subset U_0$. Apply Lemma 2 to the 2-cell F, open set V_0, and map $g_0 P$ to obtain maps α_0 and β_0 satisfying the conclusions of Lemma 2. Let $g_1 = g_0 P \alpha_0 P^{-1}$

and $f_1=\beta_0 f_0$. Note that conditions (4) and (6) of Lemma 2 imply that there is a set $N_1=\{t(1, i)\}$ of k_1+1 points in I such that $0=t(1, 0)<t(1, 1)<\cdots<t(1, k_1)=1$ and $f_1 g_1 P(t\times I)$ is a point if and only if $t\in N_1$. Define

$$F(1, i) = f_1 g_1 P([t(1, i-1), t(1, i)]\times I).$$

By applying Lemma 2 with a small enough epsilon we insure that Diam $F(1, i)<\varepsilon_0/2$ and $d(g_0, g_1)<\varepsilon_0$. There is an open set V_1 contained in $f_1(U_1)$ with exactly k_1 components $V(1, 1), \ldots, V(1, k_1)$ such that

$$F(1, i)-f_1 g_1 P(\{t(1, i-1), t(1, i)\}\times I) \subset V(1, i)$$

and Diam $V(1, i)<\varepsilon_0$.

We proceed inductively. Assume that a set $N_{n-1}=\{t(n-1, i) \mid 0=t(n-1, 0)<\cdots<t(n-1, k_{n-1})=1\}$ of $k_{n-1}+1$ points in I, an open set V_{n-1} with components $V(n-1, 1), \ldots, V(n-1, k_{n-1})$, 2-cells

$$F(n-1, i) = f_{n-1}g_{n-1}P([t(n-1, i-1), t(n-1, i)]\times I),$$

and maps g_{n-1} and f_{n-1} have been defined. For $i=1, \ldots, k_{n-1}$ apply Lemma 2 to the 2-cell $F(n-1, i)$, open set $V(n-1, i)$ and map

$$f_{n-1}g_{n-1}P\,|\,[t(n-1, i-1), t(n-1, i)]\times I$$

and obtain maps α^i_{n-1} and β^i_{n-1}. The maps α^i_{n-1} $(i=1, \ldots, k_{n-1})$ are pieced together to obtain a homeomorphism α_{n-1} from I^2 onto I^2 such that

$$\alpha_{n-1}|[t(n-1, i-1), t(n-1, i)]\times I = \alpha^i_{n-1}.$$

Let $g_n=g_{n-1}P\alpha_{n-1}P^{-1}$, and define the map f_n so that

$$f_n|E^3-f^{-1}_{n-1}(V_{n-1}) = f_{n-1}|E^3-f^{-1}_{n-1}(V_{n-1})$$
$$f_n|f^{-1}_{n-1}(V(n-1, i)) = \beta^i_{n-1}f_{n-1}|f^{-1}_{n-1}(V(n-1, i)).$$

Conditions (4) and (6) of Lemma 2 imply that there is a set $N_n=\{t(n, i)\}$ of k_n+1 points in I such that $0=t(n, 0)<\cdots<t(n, k_n)=1$, and $f_n g_n P(t\times I)$ is a point if and only if $t\in N_n$. Define

$$F(n, i) = f_n g_n P([t(n, i-1), t(n, i)]\times I).$$

By applying Lemma 2 with a small enough epsilon we insure that Diam $F(n, i)<\varepsilon_{n-1}/2$ and

(7) $d(g_{n-1}, g_n)<\varepsilon_{n-1}$.

There is an open set V_n contained in $f_n(U_n)$ with exactly k_n components $V(n, 1), \ldots, V(n, k_n)$ such that Diam $V(n, i)<\varepsilon_{n-1}$, $F(n, i)\subset V(n, i)\cup f_n g_n P(\{t(n, i-1), t(n, i)\}\times I)$, and

(8) $V(n, i)\subset V(n-1, j)$ for some j.

We further suppose that

(9) $\sum_n^\infty \varepsilon_i$ is so small that if $d(x_1, x_2)>1/n$ for $x_1, x_2\in\Delta_2$ then

$$d(g_n(x_1), g_n(x_2)) > 2\sum_n^\infty \varepsilon_i.$$

We also have that

(10) $d(f_n, f_{n+1}) < \varepsilon_{n-1}$ if $n \neq 0$,

(11) $f_n | E^3 - U_n = f_{n+1} | E^3 - U_n$,

(12) $f_n | E^3 - g_n P(N_n \times I)$ is a homeomorphism onto $E^3 - f_n g_n P(N_n \times I)$,

(13) $N_1 \subset N_2 \subset \cdots$ and $\bigcup N_i$ is dense in I,

(14) $f_n g_n P | N_n \times I = f_{n+1} g_{n+1} P | N_n \times I$,

(15) $f_n g_n P(t \times I)$ is a point if $t \in N_n$,

(16) $f_n g_n P(t \times I) \cap f_n g_n P(r \times I) = \varnothing$ if $t, r \in I$ and $t \neq r$,

(17) $\operatorname{Diam} f_n g_n P(t \times I) < \varepsilon_{n-1}$ if $t \in I$,

(18) if $t, r \in I - \bigcup N_i$ and $t \neq r$, then there are numbers n, i and j with $j \neq i$ such that $f_n g_n P(t \times I) \subset V(n, i)$, $f_n g_n P(r \times I) \subset V(n, j)$,

(19) $f_{n+1} f_n^{-1}(V(n, i)) \subset V(n, i)$.

We let $g = \text{limit } g_n$ and $f = \text{limit } f_n$. That g and f are maps follows from (6), (7), and (10). It follows from (6), (7), (9) and [7, Theorem 7] that g is a homeomorphism of Δ_2 onto F such that $d(g_0, g) < \varepsilon$. Conditions (11) and (12) imply that $f | E^3 - F$ is a homeomorphism onto $E^3 - f(F)$ and conditions (8) and (13)–(19) imply $fg\pi^{-1} = h$ is a homeomorphism of Δ_1 onto $f(F)$.

5. **Applications to cubes with handles.** In this section we extend the results of §3 to embeddings of cubes with handles.

THEOREM 10. *Suppose that H_n is a cube with n handles in the interior of a 3-manifold M, and U is an open subset of M containing H_n. Let D_n be a disk with n holes, and let π_1 be the projection of $D_n \times I$ onto D_n. Then there exist a map f of M onto itself, a homeomorphism h of D_n onto $f(H_n)$, and a homeomorphism g of $D_n \times I$ onto H_n such that*

(1) f is a homeomorphism of $M - H_n$ onto $M - f(H_n)$,

(2) $f | M - U = \text{identity}$,

(3) $fg = h\pi_1$.

The same techniques used to prove Theorem 1 may be reapplied to prove Theorem 10.

THEOREM 11. *Suppose D_n is a disk with n holes in the interior of a 3-manifold M, and U is an open subset of M containing D_n. Then there exists a map f of M onto itself such that*

(1) f is a homeomorphism of $M - D_n$ onto $M - f(D_n)$,

(2) $f | M - U = \text{identity}$,

(3) $f(D_n)$ is a wedge of n simple closed curves.

Furthermore, f may be obtained so that the preimage of a point of $f(D_n)$ is either an arc or a $2n$-frame.

Proof. Let $J_0, J_1, \ldots, J_n$ be the boundary components of D_n. Using Lemma 0 we find a tame n-frame G' in D_n such that $G' \cap \operatorname{Bd} D_n = \operatorname{Bd} G'$, $\operatorname{Bd} G' \subset J_0$, and each component of $D_n - G'$ contains (exactly) one of the curves $J_1, \ldots, J_n$. Then G' is contained in a tame $2n$-frame G in D_n such that $G \cap \text{Bd } D_n = \text{Bd } G$ and each of the boundary curves $J_1, \ldots, J_n$ meets one of the arcs of $G - G'$.

Since G is tame, there exists a map f_1 of M onto itself such that $f_1 | M - U =$ identity, f_1 takes G to a point, and f_1 is a homeomorphism of $M - G$ onto $M - f_1(G)$. Consequently, $f_1(D_n)$ is the union of pinched annuli $A_1, \ldots, A_n$ joined at a point p.

We select a tame arc α_i spanning A_i whose endpoints lie in distinct components of Bd $A_i - p$ $(i = 1, \ldots, n)$. There is a map f_2 of M onto itself such that $f_2 \; | M - U =$ identity, f_2 takes each arc α_i to a point of M, and f_2 is a homeomorphism of $M - \bigcup \alpha_i$ onto $M - f_2(\bigcup \alpha_i)$. Then $f_2 f_1(D_n)$ is the union of disks $E_1, \ldots, E_{2n}$, where the intersection of any pair of the E's is either one point or two points in the boundary of each.

Applying Theorem 2′ to each of the E's, we find that there exist a map f_3 of M onto itself, homeomorphisms g_i of Δ_2 onto E_i, and homeomorphisms h_i of Δ_1 onto $f_3(E_i)$ such that

(1) f_3 is a homeomorphism of $M - f_2 f_1(D_n)$ onto $M - f_3 f_2 f_1(D_n)$,

(2) $f_3 | M - U =$ identity,

(3) $f_3(E_j) \cap f_3(E_k) = E_j \cap E_k$, whenever $j \neq k$,

(4) $f_3 g_i = h_i \pi$ $(i = 1, \ldots, 2n)$.

The composition $f_3 f_2 f_1$ produces the required map f.

COROLLARY 12. *Suppose H_n is a cube with n handles in the interior of a 3-manifold M, and U is an open subset of M containing H_n. Then there exists a map f of M onto itself such that*

(1) *f is a homeomorphism of $M - H_n$ onto $M - f(H_n)$,*

(2) *$f | M - U =$ identity,*

(3) *$f(H_n)$ is a wedge of n simple closed curves.*

We conclude this section by showing why an arbitrary simple closed curve cannot be enlarged to a solid torus that projects onto the curve. Let J denote the simple closed curve that pierces no disk of [5].

THEOREM 13. *There is no solid torus T in E^3 such that $E^3 - T$ and $E^3 - J$ are homeomorphic.*

We require the following definitions: if T is a solid torus and J is a tame simple closed curve in Int T, the *order of T with respect to J* is the minimal number of points of intersection of a meridianal disk of T with the curve J (for all meridianal disks of T), and is denoted $O(T, J)$; if T and T' are tame solid tori with $T' \subset \text{Int } T$, then the *order of T with respect to T'*, denoted $O(T, T')$, is defined to be the order of T with respect to a center line of T'. This definition of $O(T, T')$ is independent of the choice of center line of T' [18, p. 172].

Proof of Theorem 13. Suppose T is a solid torus in E^3 such that there exists a homeomorphism h of $E^3 - J$ onto $E^3 - T$. Consider the sequence of polyhedral solid tori $T_1, T_2, \ldots$ used by Bing in the definition of J. Since the fundamental group of $h(E^3 - T_i)$ is not infinite cyclic, the closure C_i of the component of $E^3 - h(\text{Bd } T_i)$ containing T is a tame solid torus ($i \geqq 2$). Let H be a tame solid torus in Int T concentric with T, and let k be a positive integer. It is easy to show that $O(T_n, T_{n+k}) = O(C_n, C_{n+k})$ and that $O(C_n, H) > 0$ for sufficiently large integers n. Furthermore, Step 3 of [5] indicates that $O(T_n, T_{n+k}) \geqq 2^k$. It follows from the product formula of Schubert [18, p. 175] that

$$O(C_n, H) = O(C_n, C_{n+k}) \cdot O(C_{n+k}, H) \geqq 2^k \cdot O(C_{n+k}, H).$$

Consequently, if n is sufficiently large, $O(C_n, H)$ must be infinite. This is impossible.

QUESTION. Does there exist an arc A in E^3 such that for each 3-cell K in E^3, the complement of K is not homeomorphic to the complement of A?

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